2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

Adaptive Sliding Mode Control with Unknown Control Direction

C. Ton, S. S. Mehta, Z. Kan

Abstract—**This paper considers an adaptive terminal sliding mode control for systems with unknown control direction and non-vanishing disturbance. As opposed to the previous treatments of systems with unknown control direction and exogenous disturbance, where the control gain is designed with large magnitude to compensate for the worst case scenarios, this work presents a control structure that adjusts the input gain to the minimal value to maintain the sliding motion. The presented control structure enables both the sliding surface and the state to reach the equilibrium in finite time. Simulation results are provided to show the efficacy of the developed control structure.**

I. Introduction

Sliding mode control (SMC) has remained a popular control technique over the past decades due to its robustness and simplicity. Classical SMC has shown that the sliding surface has finite time stability, but the states generally reach the equilibrium exponentially [1]. Extending the finite time convergence property in the sliding surface to the state variables, terminal sliding mode control (TSMC) has been developed in [2]–[4]. One of the setbacks of TSMC is the singularity on the sliding surface as the state approaches the equilibrium. In [5], Wu *et al.* propose a transformation of the system trajectory into a prescribed set that avoids the singularity issue. In [6], Feng designs a nonsingular terminal sliding surface that is singularity-free while retaining finite convergence time property. Revisiting TSMS, Feng [3] shows that the singularity issue can be overcome by introducing the saturation function into the control law. Although TSMC has faster convergence properties than classical SMC, the challenges of TSMC with gain adaptation for chattering attenuation has to be addressed.

One of the disadvantages of SMC is the chattering phenomenon due to the high frequency switching on the sliding surface. In order to mitigate the chattering effect, supertwisting [7] and gain adaptation algorithms [8] have been proposed. Supertwisting is a continuous variable structure control method that preserves the disturbance rejection property of SMC. This control technique has been integrated with adaptive algorithm to reduce the overestimation of gain requirement to further reduce chattering [9]–[11]. In addition to continuous algorithms, discontinuous control input with gain adaption is also considered to alleviate chattering [12]–[15]. In [12], [13], Plestan *et al.* consider a gain adaptation based on the distance of the system to the surface. In [14], [16], Bartolini *et al.* propose an adaption algorithm that increases the gain until sliding mode occurs and decreases the gain until sliding mode is lost by checking at a fixed number of sampling instances. To the best of the authors' knowledge, minimal gain adaption algorithm for chattering alleviation for systems with unknown control direction has not been considered.

The challenges of unknown control direction is initially addressed by the Nussbaum gain [17]. This work has been extended to cascade systems [18] and incorporated with iterative methods to compensate for the lack of knowledge in the sign direction [19]. In addition to the Nussbaum gain, the monitoring function is also used to detect the correct sign in the input matrix [20]. This concept is implemented on output feedback system [21] and has been verified experimentally [22]. Minimum seeking Lyapunov function is another technique also used to handle systems with sign uncertainty [23]. Robust control has also been modified to cope with systems without apriori knowledge of the control direction. In [24], Kaloust *el al.* develop an online algorithm with Lyapunov functions to detect the sign changes in the input matrix. In [25], [26], Bartolini *et al.* use a suboptimal algorithm, which tests whether the magnitude of the sliding surface is decreasing or increasing, to adapt to the sign uncertainty. In [27], Drakunov overcomes the sign uncertainty by dividing the state space into cells with alternating control values, which allows the sliding motion to occur at a constant. This sliding motion has a dynamic compensator that enables the state to reach the equilibrium point.

The contribution of this paper is an adaptive terminal sliding mode control structure that achieves finite time convergence to the origin in the presence of time-varying unknown control direction and nonvanishing disturbance. In contrast to the previous work in [27], where the control gain is based upon the estimated upperbound of the disturbance, this work considers an active adaption of the magnitude control input to a minimum value that is sufficient to maintain the sliding motion. Unlike asymptotic [18] or exponentially stability [27], the developed control structure in this paper achieves finite time convergence with terminal sliding surface in the presence of sign uncertainty and exogenous disturbances. Simulation results are provided to show the efficacy of the control structure.

II. Background

The basic principle of unknown control direction can be briefly summarized in the following Lemma [28].

Lemma 1: Let the first order system be defined as

$$\dot{x} = f(t) + bu$$

where $f(t) \in \mathbb{R}$ is the disturbance, $b \in \mathbb{R}$ is the unknown input scalar, and the control input $u(t)$ is designed as

$$u = M sgn \sin \frac{\pi \tilde{s}}{\varepsilon}$$

C. Ton is with AFRL, Kirland Air Force Base, NM. S. S. Mehta is with the University of Florida - REEF. Z. Kan is with the University of Iowa. This research is supported in part by a grant from the AFRL Mathematical Modeling and Optimization Institute contracts #FA8651-08-D-0108/042-043-049. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the funding agency.

978-1-5090-2873-3/17/$31.00 ©2017 IEEE

where $\varepsilon \in \mathbb{R}^+$ is a constant, $M \in \mathbb{R}^+$ is a constant or a positive function, the hypersurface $\tilde{s}(t)$ is defined as

$$\tilde{s} = s + \lambda \int sgn(s)dt \tag{1}$$

with $s(t) = x$ is defined as the sliding surface, and $\lambda \in \mathbb{R}^+$. When the gain M is designed to satisfy the inequality $|Mb| > |f| + \lambda$, then the surface $s(t)$ goes to zero in finite time.

Proof. Taking time derivative of the hypersurface $\tilde{s}(t)$ yields

$$\dot{\tilde{s}} = f(t) + bMsgn\sin\frac{\pi\tilde{s}}{\varepsilon} + \lambda sgn(s) \tag{2}$$

When the gain M is designed such that $|Mb| > |f| + \lambda$, it is obvious that the sign of $bMsgn\sin\frac{\tilde{s}}{\varepsilon}$ is dominant in (2). In the neighborhoods of the point where

$$\tilde{s} = k\varepsilon \tag{3}$$

for $k = 0, \pm 2, \pm 4, \ldots$, the following is obtained:

$$sign\left[\sin\left(\tilde{s}\right)\right] = sgn(\tilde{s} - k\varepsilon)$$

and for $k = \pm 1, \pm 3, \ldots$, the following is obtained:

$$sign\left[\sin\left(\tilde{s}\right)\right] = -sgn(\tilde{s} - k\varepsilon)$$

Thus, sliding mode will occur on one of the manifolds in (3) for any sign of bM. In fact, sliding mode occurs where $\tilde{s} = \zeta$ after some moment of time, and $\zeta \in \mathbb{R}$ is a constant. After differentiating (1), the following equality is obtained

$$\dot{s} = -\lambda sign(s) \tag{4}$$

Thus (4) guarantees that the manifold $s(x) = 0$ is reached in finite time. ■

One of the challenges associated with designing a large gain M in (2) is the over estimation of the disturbance $f(t)$. When $f(t)$ is time varying, the magnitude M can be adaptively decreased as close as possible to the magnitude of $f(t)$ to alleviate chattering. To achieve this objective, we present an adaptive terminal sliding mode control structure that actively adjusts the magnitude of control gain to match the magnitude of the unknown disturbance in the presence of unknown control direction.

III. Problem Formulation

Consider an uncertain system modeled as a double integrator subjected to nonvanishing disturbance

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(t) + b(t)u \end{aligned} \tag{5}$$

where $x = [x_1\, x_2]^T \in \mathbb{R}^2$ is the state, $b(t) \in \mathbb{R}$ is the unknown time-varying input scalar, $f(t) \in \mathbb{R}$ is the nonvanishing disturbance, and $u(t) \in \mathbb{R}$ is the control input.

Assumption 1: The input matrix $b(t)$ and its derivative can be lower and upper bounded almost everywhere as

$$\underline{b} \leq |b| \leq \bar{b} \quad \underline{\dot{b}} \leq \left|\dot{b}\right| \leq \bar{\dot{b}}$$

where $b, \bar{b}, \underline{\dot{b}}, \bar{\dot{b}} \in \mathbb{R}$ are positive constants.

Assumption 2: The frequency that $b(t)$ changes in sign is low.

Assumption 3: The nonvanishing disturbance $f(t)$ and $\dot{f}(t)$ are bounded by known constants $\underline{f}, \bar{f}, \bar{\dot{f}} \in \mathbb{R}^+$

$$\underline{f} < |f(t)| \leq \bar{f} \quad |\dot{f}(t)| \leq \bar{\dot{f}}$$

The objective of this paper is to design an adaptive robust control algorithm that guarantees finite time stability of the system (5) in the presence of time-varying sign uncertainty in the control input and nonvanishing disturbance.

IV. Sliding Surface Design

The terminal sliding surface is designed as

$$s = x_2 + \alpha|x_1|^{p/q} sgn(x_1) \tag{6}$$

where $\alpha \in \mathbb{R}^+$ is a constant, $p, q \in \mathbb{R}^+$ are integers that satisfy the inequality $.5 \leq p/q < 1$. To compensate for the sign uncertainty, the hyper sliding surface is designed as

$$\tilde{s} = s + \lambda \int \sigma^{m_1/m_2}(s)\, d\tau \tag{7}$$

where σ is the saturation function, and $m_1, m_2 \in \mathbb{R}^+$ are odd integers that satisfy the inequality $0 < m_1/m_2 < 1$. Denoting $\sigma_s(t) = \sigma(s)$, the function σ_s can be bounded by a constant $|\sigma_s| \leq \sigma_\lambda \in \mathbb{R}^+$. Taking the time derivative of the surface $s(t)$ and hypersurface $\tilde{s}(t)$ along (5) yields the following equalities

$$\begin{aligned} \dot{s} &= \dot{x}_2 + \alpha\frac{p}{q}|x_1|^{p/q-1}|x_2 \\ \dot{\tilde{s}} &= \dot{s} + \lambda\sigma_s^{m_1/m_2} \end{aligned} \tag{8}$$

Substituting the dynamics in (5) and the surface $\dot{s}(t)$ into (8), the open-loop system can be expressed as

$$\dot{\tilde{s}} = bu + \lambda\sigma_s^{m_1/m_2} + \alpha\frac{p}{q}|x_1|^{p/q-1}x_2 + f \tag{9}$$

We present two control structures. The first algorithm contains no adaptation, and the magnitude of the control gain is designed based on the maximum amplitude of the disturbance. The second control algorithm is an adaptive algorithm, where the magnitude of the control input adjusts itself to the magnitude of the unmeasurable and nonvanishing disturbance, consequently, reducing the required control input effort.

V. Control Development

A. Terminal Sliding Mode with Unknown Control Direction

Based on subsequent analysis, the first control algorithm without gain adaptation is designed as

$$u = \left((k(t) + |\sigma_h|)\underline{b}^{-1} + \lambda_1\right)\Psi \tag{10}$$

where the functions $\sigma_h(t)$ is defined as

$$h(t) = \alpha\frac{p}{q}|x_1|^{p/q-1}x_2 \quad \sigma_h(t) = \sigma(h(t))$$

and $k(t) \in \mathbb{R} > \bar{f}$ is a positive function or constant, $\sigma(\cdot)$ is the saturation function with the upper and lower bound given as

$|\sigma_h| \leq \sigma_\mu \in \mathbb{R}^+$, $\frac{\lambda\sigma_\lambda}{\underline{b}} < \lambda_1 \in \mathbb{R}^+$, and the switching surface $\Psi(t)$ is expressed as

$$\Psi = sgn\left(\sin\frac{\pi\tilde{s}}{\varepsilon}\right)$$

Substituting the control input $u(t)$ into (9) yields

$$\dot{\tilde{s}} = Mb\Psi + \lambda\sigma_s^{m_1/m_2} + h + f \tag{11}$$

It is obvious that sliding mode occurs in finite time when $|M\underline{b}| > |h + f| + \lambda$, where the function $M(t)$ is given as

$$M = (k(t) + |\sigma_h|)\,\underline{b}^{-1} + \lambda_1.$$

Once sliding mode occurs, $\tilde{s}(t)$ approaches a constant and $s(t) \to 0$ in finite time, i.e.,

$$\dot{s} = -\lambda\sigma_s^{m_1/m_2}.$$

When $s = 0\ \forall t \geq t_r \geq 0$ and the motion of the state $x(t)$ is restricted to the sliding surface, the time that $x(t)$ reaches the origin can be calculated as [3]

$$t_s = \frac{q}{\alpha(q-p)}|x_1|^{1-p/q}(t_r). \tag{12}$$

We are ready to state the first result.

Theorem 1: The control input in (10) is sufficient to drive the sliding surface $s(t)$ and state $x(t)$ to zero in finite time, provided that $\lambda_1 > \lambda\sigma_\lambda/\underline{b}$ and $k(t) > |f(t)|$.

Proof. Based on [3], the following properties are observed

1) $|x_1^{p/q-1}|x_2 \to \infty$ as $x_1 \to 0$, and $x_2 \neq 0$;
2) $|x_1^{p/q-1}|x_2 \neq 0$ for $x_2 \neq 0$;
3) $sgn(x_2|x_1^{p/q-1}|) = sgn(x_2)$ for $x_2 \neq 0$;
4) $\lim_{(x_1,x_2)\to(0,0)} |x_1^{p/q-1}|x_2 = 0$ along the surface $s = x_2 + \alpha|x_1|^{p/q}sgn(x_1) = 0$;
5) the term $|x_1^{p/q-1}|x_2$ is singular on the x_2-axis but the point $(0,0)$ is nonsingular.

Based on the properties above, consider the two sets

$$\mathcal{A} = \{(x_1,x_2)\,||h(x_1,x_2)| \leq \sigma_\mu\}$$
$$\mathcal{B} = \{(x_1,x_2)\,||h(x_1,x_2)| > \sigma_\mu\}$$

In the sets $\mathcal{A}$ and $\mathcal{B}$, the surface $s = 0$ always lies in the set $\mathcal{A}$. The intersection of the sets $\mathcal{A}$ and $\mathcal{B}$ is only the point $(x_1,x_2) = (0,0)$, where $\lim_{(x_1,x_2)\to(0,0)} x_2|x_1|^{p/q-1}sgn(x_1) = 0$ along the surface $s = 0$. Thus, along the surface $s = 0$, there is no singularity.

Consider the set $\mathcal{B}$ and the Lyapunov function candidate

$$V = \frac{1}{2}\tilde{s}^2$$

and its time derivative along the trajectory in (5)

$$\dot{V} = \tilde{s}\left(Mb\Psi + h + f + \lambda\sigma_s^{m_1/m_2}\right)$$

Dividing $\mathcal{B}$ into two sets

$$\mathcal{B}_1 = \left\{(x_1,x_2)\}|\dot{V} = \tilde{s}\dot{\tilde{s}} < 0\right\}$$
$$\mathcal{B}_2 = \left\{(x_1,x_2)\}|\dot{V} = \tilde{s}\dot{\tilde{s}} > 0\right\}$$

Without loss of generality, let the state $x(t)$ enter $\mathcal{B}$ from $\mathcal{A}$. The state $x(t)$ might enter $\mathcal{B}_1$, but it cannot stay in there for long since $h(t)$ increases, and $x(t)$ enters $\mathcal{B}_2$ since the magnitude $|Mb|$ cannot dominate $h(t)$.

In the region where $h(t) > 0$ and $h(t) \in \mathcal{B}_2$, sliding mode is violated and the hypersurface $\tilde{s}(t)$ is no longer switching on a constant. The control input sign Ψ oscillates between -1 and 1 as $x(t) \in \mathcal{B}_2$, but it does not attach itself to a constant ζ (see Lemma 1), therefore, the surface $s(t)$ is not decaying and $\tilde{s}\dot{\tilde{s}}, s\dot{s} > 0$. By geometry and the system in (5), the trajectory of $x_1(t)$ evolves over time with the following equality

$$x_1 = x_1(0) + \int_0^t x_2(t)dt \tag{13}$$

Due to the fact that $x_2(t) > 0$ when $h(t) > 0$, equation (13) shows that $x_1(t)$ increases monotonically in a single direction and leaves the set $\mathcal{B}_2 \cap (h(t) > 0)$ and enters $\mathcal{A}$.

In the region where $h(t) < 0$ and $h(t) \in \mathcal{B}_2$, sliding mode is also violated due to insufficient gain. The control input sign oscillates ± 1, but Ψ does not attach itself to a constant ζ, and $\tilde{s}\dot{\tilde{s}}, s\dot{s} > 0$. Similarly, when $x_2(t) < 0$ and $h(t) < 0$, as in (13), the state $x_1(t)$ increases monotonically in a single direction until it crosses the boundary between $\mathcal{B}_2$ and $\mathcal{A}$, i.e $x(t)$ enters $\mathcal{A}$.

Based on previous arguments, the state $x(t)$ will not stay in the set $\mathcal{B}$ forever, instead it enters the set $\mathcal{A}$ in finite time. Consequently, sliding mode occurs and $\tilde{s}(t)$ reaches a constant in finite time. This implies that $s(t)$ reaches zero in finite time and maintain $s = 0$ afterward. ■

To keep the state $x(t)$ on the surface $s(t) = 0$, $x_2 = -\alpha|x_1|^{p/q}sgn(x_1)$ from (6) and $x_2 = -\sigma_\mu(q/(p\alpha))|x_1|^{1-p/q}$ from the set $\mathcal{A}$. Thus, the condition for choosing σ_μ is given as

$$\alpha^2(p/q)|x_1|^{2p/q-1} \leq \sigma_\mu < \bar{f} + \lambda + k + \bar{b}u_{max}$$

where the bounded constant $\bar{f}$ is given in Assumption 3, and $u(t) < u_{max} \in \mathbb{R}^+$. When p/q is chosen as $1/2$, the inequality becomes $\alpha^2(p/q) \leq \sigma_\mu < \bar{f} + \lambda + k + \bar{b}u_{max}$, and the lowerbound is independent of x_1. Thus, σ_μ can be tuned according to α^2.

B. Gain Adaptation

The presented control algorithm in the previous section is designed for worst case scenario, where the minimal value of the gain $M(t)$ is larger than the upperbound on $a(t) = h + f + \lambda\sigma_s^{m_1/m_2}$. Since $a(t)$ is time varying, the gain $M(t)$ can be decreased to reduce the chattering amplitude. Therefore, the objective of this section it to design an adaptive gain $k_1(t) \in \mathbb{R}^+$ that can be reduced to a value that is as close as possible to the magnitude of $a(t)$ while maintaining the sliding motion.

The adaptive gain control structure is proposed as

$$u = -k_1(t)\Psi. \tag{14}$$

When the condition $k_1(t) > |a(t)|$ holds, sliding mode occurs on the surface $\tilde{s} = \zeta$ within some time interval, $\dot{\tilde{s}} = 0$ and the right hand side of (9) is replaced by the equivalent control

$$u_{eq} = -\frac{a(t)}{b}$$

Setting equation (9) equal to zero and solving for the control input, the equivalent control u_{eq} can be rewritten as

$$\dot{s} = 0 : k_1(t)\Psi_{eq} = \frac{a}{b} \tag{15}$$

The equivalent control u_{eq} usually cannot be implemented because the disturbance $f(t)$ is unknown. Moreover, the equivalent control describes the average effect of the high frequency switching of the discontinuous controller in (14) on the system (5). This average effect of u_{eq} can be estimated through a low pass filter with the high frequency term Ψ. Details on the concept of equivalent control and its implementation can be found in [29]. The equivalent control Ψ_{eq} can be estimated as

$$\tau\dot{y} = \Psi - y$$

and the output $y(t)$ yields the estimate of the equivalent control Ψ_{eq}, which is in the range $(-1, 1)$, and $\tau > 0$ is a small constant.

The gain $k_1(t)$ has to satisfy the condition $k_1(t) > |a(t)/b|$ for sliding mode to occur, and the amplitude of the control input $u(t)$ is dependent upon the gain $k_1(t)$. Since $a(t)$ is time varying, the objective is to decrease the chattering amplitude and allows $k_1(t)$ to approach $a(t)/(b\beta)$, where $\beta \in (0, 1)$ and β is close to 1. The result is an adaptive algorithm that constantly tunes the gain $k_1(t)$ to approach as close as possible to the magnitude of the time varying function $a(t)/(b\beta)$ to mitigate the chattering phenomenon.

Remark 1: With a known lowerbound on $\underline{f} \leq |f(t)|$, we can design $\sigma_\mu + \lambda\sigma_\lambda^{m_1/m_2} < \underline{f}$. When $x(t) \in \mathcal{B}_2$, that is when $s\dot{s} > 0$, we observe that $x_1(t)$ increases monotonically in a single direction and leaves the set $\mathcal{B}_2$ and enters $\mathcal{A}$. When $x(t) \in \mathcal{A}$, using the fact that $|\underline{f}| - \sigma_\mu - \lambda\sigma_\lambda^{m_1/m_2} > 0$, then $|a(t)| > \underline{a} > 0$ can be satisfied through properly chosen parameters $\lambda, \sigma_\lambda^{m_1/m_2}, \sigma_\mu$.

Theorem 2: The gain adaption of $k_1(t)$, designed as

$$\begin{aligned}\dot{k}_1 &= \rho k_1 \text{sgn}(\delta) - \eta\left[k_1 - \bar{k}_1\right]_+ + \eta\left[\underline{k}_1 - k_1\right]_+ \\ &= \rho k_1 \text{sgn}(\delta) - \eta_1 + \eta_2\end{aligned} \tag{16}$$

where $\eta, \rho, \bar{k}_1, \underline{k}_1 \in \mathbb{R}^+$, the functions $\delta(t)$ and $[z]_+$ are given as

$$\delta = |\Psi_{eq}| - \beta \quad \beta \in (0, 1)$$
$$[z]_+ = \begin{cases} 1 & \text{if} \quad z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and the constants η and ρ are designed with the lowerbounds

$$\eta > \rho\bar{k}_1 \quad \rho > 0,$$

allows the error term $\delta(t)$

$$\delta = \frac{|a(t)|}{|b|k_1(t)} - \beta \tag{17}$$

to reach zero in finite time in the presence of unknown control direction and unmeasurable nonvanishing disturbance; thereby, $|b(t)k(t)|$ approaches a small vicinity of the magnitude of $a(t)$ that is sufficiently large to maintain sliding mode.

Proof. The bounds of the positive function k_1 in (16), $\underline{k_1}(t) < k_1(t) < \bar{k}_1(t)$, are designed as

$$\bar{k}_1(t) = \lambda_1 + \sigma_\mu + \bar{f} \quad \underline{k}_1 = \lambda_1 + \sigma_\mu$$

where $\sigma_\mu \in \mathbb{R}$ is a positive constant. The upperbound and lowerbound of $k_1(t)$ is designed to be larger than $\lambda\sigma_\lambda^{m_1/m_2}$ to dominate the term $\lambda\sigma_s^{m_1/m_2}$ in (9). The bounds on the gain $k_1(t)$ also include the term σ_μ to ensure sufficiently high magnitude to guarantee the occurrence of sliding mode. Based on the lowerbound $\underline{k}_1$, the magnitude of $k_1(t)$ is already sufficient to compensate for the term $\lambda\sigma_s^{m_1/m_2}$ and σ_h in (11), therefore the time derivative $\dot{h}(t)$ and $d(\lambda\sigma_s^{m_1/m_2})/dt$ are not taken into account in the subsequent analysis. The only term remains that needs adaptation is the disturbance $f(t)$, therefore, the time derivative of $a(t)$ can be upperbounded as $|\dot{a}(t)| < A \in \mathbb{R}^+$.

Consider the Lyapunov function candidate

$$V = \frac{1}{2}\delta^2.$$

Assume that the adaption process happens when $k_1(t) \in (\underline{k}_1, \bar{k}_1)$, which means that $|a(t)| > \beta\underline{k}_1$, and the terms η_1 and η_2 are zero. Taking the time derivative of $V(t)$ yields

$$\dot{V} = \delta\frac{d}{dt}\left(\frac{|a|}{k_1|b|}\right)$$

Expanding $\dot{V}(t)$ through chain rules gives the expression

$$\begin{aligned}\dot{V} &= \delta\left(\frac{\dot{a}\text{sgn}(a)}{k_1|b|} - \frac{|a|\,\dot{b}}{k_1 b^2} - \frac{|a|\,\dot{k}_1}{k_1^2|b|}\right) \\ &= \frac{\delta}{k_1|b|}\left(\dot{a}\text{sgn}(a) - |a|\,\frac{\dot{b}}{|b|} - \frac{|a|}{k_1}\left(\rho k \text{sgn}(\delta) - \eta_1 + \eta_2\right)\right) \\ &= \frac{\delta}{k_1|b|}\left(\dot{a}\text{sgn}(a) - |a|\,\frac{\dot{b}}{|b|} - |a|\,\rho\text{sgn}(\delta)\right) \\ &\leq \frac{|\delta|}{k_1\,|b|}\left(A + |a|\,\frac{\bar{\dot{b}}}{|\underline{b}|} - |a|\,\rho\text{sgn}(\delta)\right)\end{aligned} \tag{18}$$

In order to make the inequality in (18) negative definite, the positive constant ρ is designed as

$$\rho = \underline{b}^{-1}\dot{b}^+ + |\underline{a}^{-1}|\bar{A}$$

where $\bar{A} \in \mathbb{R}^+$ is designed to satisfy $\bar{A} > A$, and $\underline{a} \leq |a| \leq \bar{a}$. It can be concluded that $\dot{V}(t)$ is negative definite

$$\begin{aligned}\dot{V} &\leq -\frac{|\delta|}{k_1\,|b|}\left(|a|\frac{\bar{A}}{|\underline{a}|} - A\right) \\ &\leq -\,|\delta|\,\varkappa\end{aligned} \tag{19}$$

where $\varkappa > 0$. The inequality in (19) can be rewritten as

$$\dot{V} \leq -\varkappa\,(2V)^{1/2}$$

It follows that $V(t)$ goes to zero in finite time

$$t_f = \frac{2}{2^{1/2}\underline{\varkappa}}V_0^{1/2} \tag{20}$$

where $\underline{\varkappa}$ is the lowerbound of $\varkappa$ obtained as

$$\underline{\varkappa} = \frac{\bar{A} - A}{\bar{k}_1 \bar{b}}.$$

■

After the adaption process, $\delta(t > t_f) = 0$, the gain $k_1(t)$ is given as $k_1 = |a(t)/b\beta|$. During the adaption process, if $|a(t)| < \beta \underline{k}_1$, then $k_1(t)$ decreases until it reaches $k_1(t) = \underline{k}_1$. Due to the nature of the time-varying function $f(t)$, if $|a(t)|$ increases then it can result in $|a(t)| = \beta \underline{k}_1$. Similarly, for the motion in $k(t) \in [\underline{k}_1, \bar{k}_1)$ with the initial condition of $\delta(t_f) = 0$, the relation $k_1 = |a(t)/b\beta|$ is also maintained. When $k = \bar{k}$, the control gain is large enough to dominate the disturbance $f(t)$, sliding mode occurs and $\delta(t) = 0$ is reached in finite time.

Consider the case where sliding mode has not occurred in the set $\mathcal{A}$ and $|\Psi_{eq}| = 1$, the maximum time interval for $k(t)$ to increase from $\underline{k}_1$ to $\bar{k}_1$ is given by

$$t_k = \frac{\bar{k}_1 - \underline{k}_1}{\rho \underline{k}_1}. \tag{21}$$

Additionally, when sliding mode has not occurred, the minimum amount of time that it takes Ψ_{eq} to transition from ± 1 to ∓ 1 is

$$t_{\tilde{s}} = \frac{\varepsilon}{|\dot{\tilde{s}}_{\max}|}$$

where

$$\dot{\tilde{s}}_{\max} = \bar{b}\bar{k}_1 + \sigma_\mu + \lambda + \bar{f}.$$

Clearly, in order to guarantee the existence of sliding mode, i.e. to ensure that $k_1(t)$ reaches $\bar{k}_1$ from $\underline{k}_1$ before Ψ_{eq} changes in sign, the constant ε is chosen to satisfy the inequality

$$\frac{\bar{k}_1 - \underline{k}_1}{\rho \underline{k}_1} < \frac{\varepsilon}{|\dot{\tilde{s}}_{\max}|}.$$

When $k_1(t) = \bar{k}_1$, sliding mode is guaranteed.

Remark 2: Let $t_b \in \mathbb{R}^+$ be the minimal time that interval that $b(t)$ switches in sign. It is clear from (20) and (21) that in order for sliding mode to occur, from assumption 2, the time inequality $t_f + t_k \ll t_b$ must be satisfied; otherwise, the sign of the unknown scalar of $b(t)$ is switching too fast for sliding mode to occur.

VI. Simulation

We present a simulation for a second order system in (5) to demonstrate the robustness of the proposed control structure. The parameters used in the simulations were chosen as

$$\begin{array}{cccc} \underline{k}_1 = .5 & \bar{k}_1 = 10 & \alpha = 1 & \lambda = 1 \\ \varepsilon = 40.2 & \rho = .1 & \beta = .9 & m_1 = 3 \\ m_2 = 5 & p = 1 & q = 2 & \sigma_\mu = .5 \end{array}$$

and $\sigma_\lambda = 1$. The control input direction $b(t)$ was given as

$$b = (|\sin(t/40)| + .5) sgn(\sin(t/50))$$

for completeness, but it was not used in the control input. The nonvanishing disturbance $f(x,t)$ was given as a sinusoidal function of time

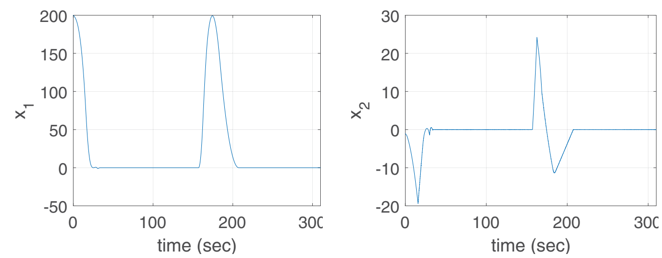
$$f(t) = \sin(t/45) + 2.1$$

The initial conditions were given as

$$k(0) = 3 \quad x_1 = 200 \quad x_2(0) = -1$$

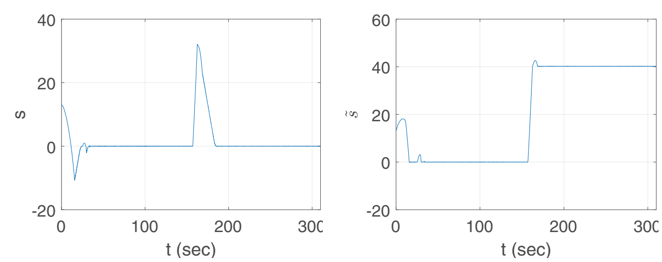
Figures 1-5 depict the position $x_1(t)$, velocity $x_2(t)$, surface $s(t)$, hypersurface $\tilde{s}(t)$, gain $k_1(t)$, control input $u(t)$, and the zoomed control input effort, respectively. In Figures 1-2, it can be seen that $\tilde{s}(t)$ goes to two constants for different time intervals, which causes the convergence of $s(t)$ and the vector $x(t)$ to zero in finite time. Around the 141^{st} second, the sign of $b(t)$ changes, which causes the hypersurface $\tilde{s}(t)$ to move to the next attractive manifold, consequently, $s(t)$ and $x(t)$ decayed to zero.

Figure 3 shows how the gain $|b|k_1(t)$ adapts to the disturbance $a(t)$ through the equivalent control Ψ_{eq}, i.e $k_1(t) = |a/b\beta|$, defining the chattering amplitude. It shows that $|b|k_1(t)$ adapts to the time-varying magnitude of $a(t)$, allows sliding mode to continue occur while lowering the chattering amplitude. From Figure 3 - 5, it is easily seen that the gain $k_1(t)$ adapts to a magnitude just large enough to compensate for $a(t)$ and maintain sliding mode.



(a) State $x_1(t)$ over time. (b) State $x_2(t)$ over time.

Fig. 1: State $x(t)$ evolution over time.



(a) Surface $s(t)$ over time. (b) Hypersurface $\tilde{s}(t)$ over time.

Fig. 2: Sliding surface evolution over time.

VII. Conclusion

In this paper, we present an adaptive terminal sliding mode control for systems with unknown control direction. As opposed to previous approach where the magnitude of the control input is designed to be always larger than the upperbound of the disturbance, we propose an active adaptation that minimizes magnitude of the control gain to alleviate chattering. Moreover, the system is finite time stable in the presence of nonvanishing disturbance and uncertainty in the control direction. Simulation results are provided to demonstrate the performance of the developed control structure.

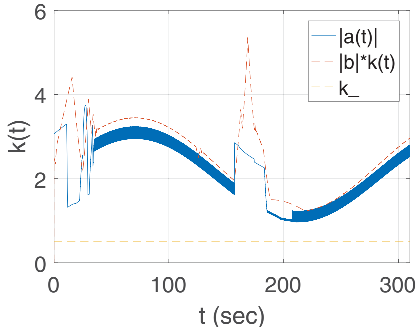


Fig. 3: The gain $k(t)$ compared with the disturbance $a(t)$.

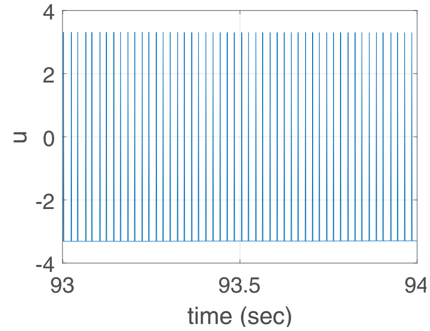


Fig. 5: Zoomed in control input $u(t)$.

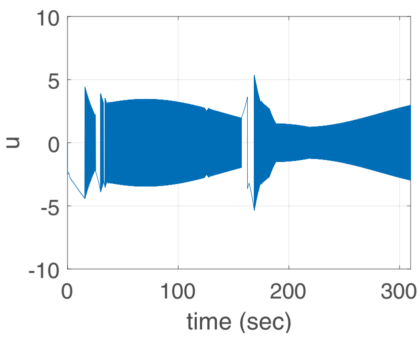


Fig. 4: $u(t)$.

REFERENCES

[1] V. Utkin, “Variable structure systems with sliding modes,” *IEEE Transactions on Automatic Control*, vol. 22, no. 2, pp. 212–222, Apr 1977.

[2] S. Venkataraman and S. Gulati, “Control of nonlinear systems using terminal sliding modes,” *Journal of dynamic systems, measurement, and control*, vol. 115, no. 3, pp. 554–560, 1993.

[3] Y. Feng, X. Yu, and F. Han, “On nonsingular terminal sliding-mode control of nonlinear systems,” *Automatica*, vol. 49, no. 6, pp. 1715 – 1722, 2013.

[4] Z. Zuo, “Non-singular fixed-time terminal sliding mode control of non-linear systems,” *IET Control Theory & Applications*, vol. 9, no. 4, pp. 545–552, 2015.

[5] Y. Wu, X. Yu, and Z. Man, “Terminal sliding mode control design for uncertain dynamic systems,” *Systems & Control Letters*, vol. 34, no. 5, pp. 281 – 287, 1998.

[6] Y. Feng, X. Yu, and Z. Man, “Non-singular terminal sliding mode control of rigid manipulators,” *Automatica*, vol. 38, no. 12, pp. 2159 – 2167, 2002.

[7] A. Levant, “Sliding order and sliding accuracy in sliding mode control,” *International Journal of Control*, vol. 58, no. 6, pp. 1247–1263, 1993.

[8] H. Lee and V. I. Utkin, “Chattering suppression methods in sliding mode control systems,” *Annual reviews in control*, vol. 31, no. 2, pp. 179–188, 2007.

[9] Y. Shtessel, M. Taleb, and F. Plestan, “A novel adaptive-gain supertwisting sliding mode controller: Methodology and application,” *Automatica*, vol. 48, no. 5, pp. 759 – 769, 2012.

[10] J. A. Moreno, D. Y. Negrete, V. Torres-Gonzlez, and L. Fridman, “Adaptive continuous twisting algorithm,” *International Journal of Control*, vol. 89, no. 9, pp. 1798–1806, 2016.

[11] H. Alwi and C. Edwards, “An adaptive sliding mode differentiator for actuator oscillatory failure case reconstruction,” *Automatica*, vol. 49, no. 2, pp. 642–651, 2013.

[12] F. Plestan, Y. Shtessel, V. Bregeault, and A. Poznyak, “New methodologies for adaptive sliding mode control,” *International journal of control*, vol. 83, no. 9, pp. 1907–1919, 2010.

[13] ——, “Sliding mode control with gain adaptation–Application to an electropneumatic actuator,” *Control Engineering Practice*, vol. 21, no. 5, pp. 679–688, 2013.

[14] G. Bartolini, A. Levant, F. Plestan, M. Taleb, and E. Punta, “Adaptation of sliding modes,” *IMA Journal of Mathematical Control and Information*, p. dns019, 2012.

[15] V. I. Utkin and A. S. Poznyak, “Adaptive sliding mode control with application to super-twist algorithm: Equivalent control method,” *Automatica*, vol. 49, no. 1, pp. 39 – 47, 2013.

[16] G. Bartolini, A. Levant, A. Pisano, and E. Usai, “Adaptive second-order sliding mode control with uncertainty compensation,” *International Journal of Control*, pp. 1–12, 2016.

[17] R. D. Nusssbaum, “Some Remarks on a Conjecture in Parameter Adaptive Control,” *System and Control Letters*, pp. 243–246, 1983.

[18] Y. Xudong and J. Jingping, “Adaptive nonlinear design without a priori knowledge of control directions,” *IEEE Transactions on Automatic Control*, vol. 43, no. 11, pp. 1617–1621, Nov 1998.

[19] M. Yu, X. Ye, and D. Qi, “Robust adaptive repetitive learning control for a class of time-varying nonlinear systems with unknown control direction,” *Journal of Control Theory and Applications*, vol. 11, no. 3, pp. 336–342, 2013.

[20] L. Yan, L. Hsu, R. R. Costa, and F. Lizarralde, “A variable structure model reference robust control without a prior knowledge of high frequency gain sign,” *Automatica*, vol. 44, no. 4, pp. 1036 – 1044, 2008.

[21] T. R. Oliveira, A. J. Peixoto, E. V. L. Nunes, and L. Hsu, “Control of uncertain nonlinear systems with arbitrary relative degree and unknown control direction using sliding modes,” *International Journal of Adaptive Control and Signal Processing*, vol. 21, no. 8-9, pp. 692–707, 2007.

[22] A. J. Peixoto, T. R. Oliveira, and H. Liu, “Sliding mode control of uncertain systems with arbitrary relative degree and unknown control direction: Theory and experiments,” in *IEEE Conf. on Decision and Control*, Dec. 2006, pp. 4951–4956.

[23] A. Scheinker and M. Krstić, “Minimum-seeking for clfs: Universal semiglobally stabilizing feedback under unknown control directions,” *IEEE Transactions on Automatic Control*, vol. 58, no. 5, pp. 1107–1122, May 2013.

[24] J. Kaloust and Z. Qu, “Continuous robust control design for nonlinear uncertain systems without a priori knowledge of control direction,” *IEEE Transactions on Automatic Control*, vol. 40, no. 2, pp. 276–282, Feb. 1995.

[25] G. Bartolini, A. Ferrara, and L. Giacomini, “A switching controller for systems with hard uncertainties,” *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* , vol. 50, no. 8, pp. 984–990, 2003.

[26] G. Bartolini, A. Pisano, and E. Usai, “On the second-order sliding mode control of nonlinear systems with uncertain control direction,” *Automatica*, vol. 45, no. 12, pp. 2982 – 2985, 2009.

[27] S. Drakunov, U. Ozguner, P. Dix, and B. Ashrafi, “ABS control using optimum search via sliding modes,” *IEEE Transactions on Control Systems Technology*, vol. 3, no. 1, pp. 79–85, 1995.

[28] S. V. Drakunov, “Sliding mode control with multiple equilibrium manifolds,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 55, no. 1, pp. 101–108, November 1994.

[29] Y. Shtessel, C. Edwards, L. Fridman, and A. Levant, *Sliding mode control and observation*. Springer, 2014.